Exercise 15

Graph the two basic solutions along with several other solutions of the differential equation. What features do the solutions have in common?

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$$

Solution

This is a linear homogeneous ODE with constant coefficients, so its solutions are of the form $y = e^{rx}$.

$$y = e^{rx} \quad \rightarrow \quad \frac{dy}{dx} = re^{rx} \quad \rightarrow \quad \frac{d^2y}{dx^2} = r^2 e^{rx}$$

Plug these formulas into the ODE.

$$r^2 e^{rx} + 2(re^{rx}) + 2(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 + 2r + 2 = 0$$

Solve for r.

$$r = \frac{-2 \pm \sqrt{4 - 4(1)(2)}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm 2i$$

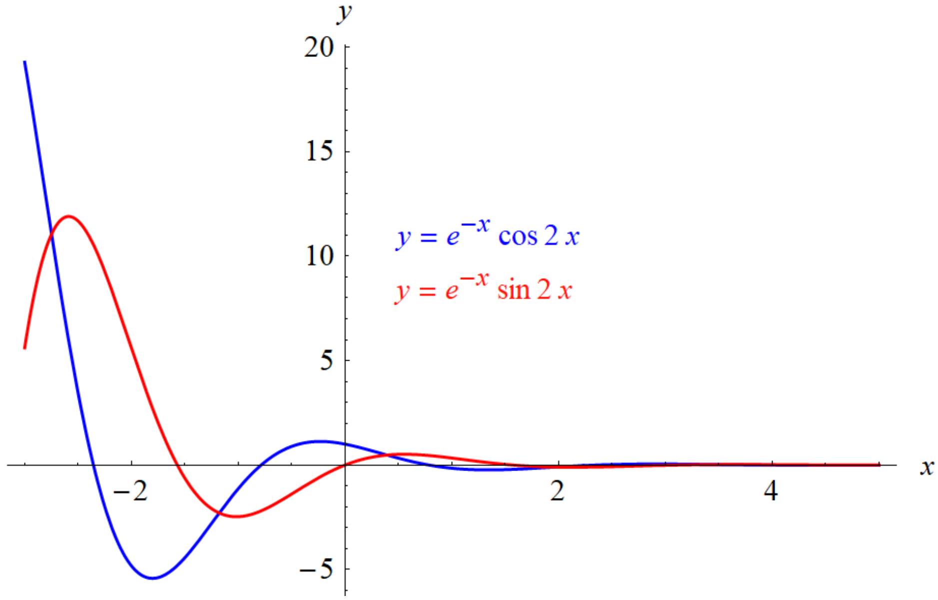
$$r = \{-1 - 2i, -1 + 2i\}$$

Two solutions to the ODE are $e^{(-1-2i)x}$ and $e^{(-1+2i)x}$. By the principle of superposition, then,

$$\begin{aligned} y(x) &= C_1 e^{(-1-2i)x} + C_2 e^{(-1+2i)x} \\ &= C_1 e^{-x} e^{-2ix} + C_2 e^{-x} e^{2ix} \\ &= e^{-x}(C_1 e^{-2ix} + C_2 e^{2ix}) \\ &= e^{-x}[C_1(\cos 2x - i \sin 2x) + C_2(\cos 2x + i \sin 2x)] \\ &= e^{-x}[(C_1 + C_2)\cos 2x + (-iC_1 + iC_2)\sin 2x] \\ &= e^{-x}(C_3 \cos 2x + C_4 \sin 2x), \end{aligned}$$

where C_1, C_2, C_3, and C_4 are arbitrary constants.

Below is a graph of these two solutions.



Both solutions blow up as $x \to -\infty$.